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Cycles in Bipartite Graphs

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For k an integer, let $G(a, b, k)$ denote a simple bipartite graph with bipartition (A, B) where $|A| = a \geqslant 2$, $|B| = b \geqslant k \geqslant 2$, and each vertex of A has degree at least k. We prove two results concerning the existence of cycles in $G(a, b, k)$.

Let k be an integer greater than one. In the following, $G(a, b, k)$ is a simple bipartite graph with bipartition (A, B), where $|A| = a \geqslant 2$, $|B| = b \geqslant k$, and each vertex of A has degree at least k. We shall prove two results on the existence of certain cycles in $G(a, b, k)$. The first result was stated as a conjecture by Sheehan [4].

THEOREM 1. *If a graph $G(a, b, k)$ satisfies $a \leqslant k$ and*

$$b \leqslant 2k - 2, \tag{1}$$

then it contains a cycle of length $2a$.

Theorem 1 is best possible in the following sense. Consider a separable graph $G(a, 2k-1, k)$, with two blocks $K_{a_1,k}$ and $K_{a_2,k}$ where $a_1 + a_2 = a \leqslant k$, and the cut vertex lies in the k-set of each block. Such a $G(a, 2k-1, k)$ clearly contains no cycle of length $2a$.

The theorem also has the following immediate corollary.

COROLLARY 1. *Suppose a graph $G(a, b, k)$ satisfies $b \leqslant 2k - 2$. If L is a subset of A containing at most k vertices, then $G(a, b, k)$ contains a cycle C such that $V(C) \cap A = L$.*

Let $f(a, k)$ be the maximum function such that if $b \leqslant f(a, k)$, then $G(a, b, k)$ contains a cycle of length $2l$ for all l, $2 \leqslant l \leqslant \min(a, k)$. It follows from Corollary 1 and the graph $G(a, 2k-1, k)$ described above that $f(a, k) = 2k - 2$ for $a \leqslant k$. It remains an open problem, however, to

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determine $f(a, k)$ for $a > k$. By Corollary 1, $f(a, k) \geqslant 2k - 2$. We also know that

$$\limsup_{a \to \infty} \left(\frac{f(a, k)}{k \sqrt{a}} \right) \leqslant 1$$

since, for p a prime greater than or equal to k, the graphs $G(p^2, pk, k)$ of [2] contain no cycles of length four. We note further that Singleton [5] has shown the existence of a graph $G(q^2 + q + 1, q^2 + q + 1, q + 1)$ which does not contain cycles of length four, is equivalent to the existence of a projective plane of order q.

In contract, Theorem 2 determines the maximum function $g(a, k)$ such that if $b \leqslant g(a, k)$, then $G(a, b, k)$ contains a cycle of length at least $2k$. For x a real number, define $\{x\}$ to be the smallest integer which is greater than or equals to x.

THEOREM 2. *If a graph $G(a, b, k)$ satisfies*

$$b \leqslant \left\{ \frac{a}{k - 1} \right\} (k - 1), \tag{2}$$

then it contains a cycle of length at least $2k$.

The fact that the function, $g(a, k) = \{a/(k - 1)\}(k - 1)$ is maximum follows from a construction given in [2].

Before proving Theorem 1, we need the following definitions. For H a subgraph of a graph G, let $V(H)$ denote the set of vertices of H. For $v \in V(G)$, let $N_H(v)$ be the set of vertices of H which are joined to v by edges of G. For $U \subseteq V(G)$, put

$$N_H(U) = \bigcup_{v \in U} N_H(v).$$

In order to simplify notation, we shall denote $V(G)$, $N_G(v)$, and $N_G(U)$ by V, $N(v)$, and $N(U)$, respectively.

If $A, B \subseteq V(G)$, let $\varepsilon(A, B)$ be the number of edges in G between the vertices of A and the vertices of B. For $v \in V$, we shall denote $\varepsilon(\{v\}, A)$ by $\varepsilon(v, A)$, and $\varepsilon(v, V)$ by $d(v)$.

Proof of Theorem 1. The proof is by contradiction. Let $G = G(a, b, k)$ be a counterexample to the theorem and

$$C = x_1 y_1 x_2 y_2 \cdots x_m y_m x_1$$

be a longest cycle in G such that $V(C) \cap A = \{x_1, x_2, \ldots, x_m\}$. Then

$$m < a \tag{3}$$

and hence we may choose $x \in A \backslash V(C)$.

Put $Q = N_{G-C}(x)$, $q = |Q|$, $R = B \backslash (V(C) \cup Q)$, and $r = |R|$. Then

$$r = b - m - q. \tag{4}$$

The proof splits into two cases, depending on the size of q.

(i) $q \geqslant k$. Choose $x_i \in V(C) \cap A$. Using (4) and (1), it follows that

$$\varepsilon(x_i, V(C) \cup R) \leqslant m + r = b - q \leqslant 2k - 2 - k = k - 2.$$

Hence each vertex of $A \cap V(C)$ is joined to at least two vertices of Q, which implies that x_1 and x_2 are joined to distinct vertices, y and y', respectively, of Q. The cycle

$$C' = x_1 y x y' x_2 y_2 x_3 \cdots x_m y_m x_1$$

contradicts the choice of C as a longest cycle in G.

(ii) $q \leqslant k - 1$. Put $S = \{x_i \in V(C) \mid y_i \in N_C(x)\}$. Clearly,

$$|S| = |N_C(x)| \geqslant k - q. \tag{5}$$

If some vertex $x_i \in S$ is joined $y \in Q$, then the cycle

$$C' = x_i y x y_i x_{i+1} y_{i+1} \cdots x_m y_m x_1 y_1 \cdots x_{i-1} y_{i-1} x_i$$

contradicts the choice of C. Hence we may assume that $\varepsilon(S, Q) = 0$. For $x_i \in S$, $\varepsilon(x_i, V(C)) \leqslant m$. Hence $\varepsilon(x_i, R) \geqslant k - m$ and, by (5), $\varepsilon(S, R) \geqslant |S|(k - m) \geqslant (k - q)(k - m)$. Using (4) and (1),

$$\begin{aligned} \varepsilon(S, R) - r &\geqslant (k - q)(k - m) - (b - q - m) \\ &\geqslant (k - q)(k - m) - (2k - 2 - q - m) \\ &= (k - q - 1)(k - m - 1) + 1 \geqslant 1, \end{aligned}$$

since $k - m - 1 \geqslant 0$, by (3). Thus there exist two vertices $x_i, x_j \in S$ which are joined to the same vertex $y \in R$, and the cycle

$$C'' = y_i x y_j x_{j+1} y_{j+1} \cdots x_i y x_j y_{j-1} x_{j-1} \cdots x_{i+1} y_i$$

contradicts the choice of C. ∎

The following lemma generalises a result of Pósa [3, Theorem 3, IV], which itself extended a result of Dirac [1, Lemma 2].

LEMMA 3. *Let u and v be distinct vertices of a 2-connected graph G. Let P be a uv-path in G and put*

$$T = \{w \in N_{G-P}(\{u, v\} \mid N(w) \subseteq V(P)\}.$$

Then there exist internally disjoint uv-paths P_1 and P_2 such that

(a) *for $i = 1$ and 2, the common vertices of P_i and P occur in the same order in both paths, and*

(b) $N_P(T \cup \{u, v\}) \subseteq V(P_1) \cup V(P_2)$.

Proof. The proof is by contradiction. Suppose that the theorem is false and let G be a counterexample with as few vertices as possible. Clearly $|V| \geqslant 4$. Pósa's result [3, Theorem 3, IV] guarantees the existence of internally disjoint uv-paths P_1 and P_2 satisfying (a), such that $N_P(\{u, v\}) \subseteq V(P_1) \cup V(P_2)$. Thus T is not empty and, without loss of generality, we may assume that there exists a vertex $w_0 \in T \cap N(u)$. Let G^* be the graph obtained from $G - \{u, w_0\}$ by adding a new vertex $u^*$and joining u^* to each vertex of $N(u, w_0\})\backslash\{u, w_0\}$.

Suppose G^* is separable. Since G is 2-connected, it follows that u^* is a cut vertex of G^*. Hence there exists a partition of $V\backslash\{u, w_0\}$ into proper subsets M_1 and M_2 such that $\varepsilon(M_1, M_2) = 0$. Clearly, either $V(P) \subseteq M_1 \cup \{u\}$ or $V(P) \subseteq M_2 \cup \{u\}$. Without loss of generality assume that $V(P) \subseteq M_1 \cup \{u\}$. Since $N(w_0) \subseteq V(P)$, it follows that $\varepsilon(M_1 \cup \{w_0\}, M_2) = 0$, which contradicts the hypothesis that G is 2-connected. Hence G^* is non-separable.

Since $|V(G^*)| = |V| - 1 \geqslant 3$, G^* is 2-connected. Let P^* be the subgraph of G^* induced by the edges of P. Then P^* is a u^*v-path in G^*. Put

$$T^* = \{w \in N_{G^*-P^*}(\{u^*, v\}) \mid N_{G^*}(w) \subseteq V(P^*)\}.$$

Then

$$T\backslash\{w_0\} \subseteq T^*. \tag{6}$$

Since $|V(G^*)| < |V|$, there exist internally disjoint u^*v-paths P_1^* and P_2^* in G^* such that

(a*) for $i = 1$ and 2, the common vertices of $P_i^*$and P^* occur in the same order on both paths, and

(b*) $N_{P^*}(T^* \cup \{u^*, v\}) \subseteq V(P_1^*) \cup V(P_2^*)$.

By (6),

$$N_P(T \cup \{u, v\}) \subseteq N_{P^*}(T^* \cup \{u^*, v\}) \cup \{u\}.$$

and hence it follows from (b*) that

$$N_P(T \cup \{u, v\}) \subseteq V(P_1) \cup V(P_2) \cup \{u\}.$$

For $i = 1$ and 2, let P_i' be the path induced in G by the edges of P_i^* and let w_i be the end vertex of P_i' corresponding to the end vertex u^* of P_i^*. Clearly P_1' and P_2' can be extended to internally disjoint uv-paths in G which satisfy conditions (a) and (b) of the lemma unless $w_1 = w_2 = w_0$. If this is the case, let z be the first vertex of P_1' or P_2' encountered in passing along P from u to v. Without loss of generality assume that $z \in V(P_1')$. By (a*),

$$(V(P_1') \cup V(P_2')) \cap V(P[u, z]) = \{z\}.$$

Putting $P_1 = P[u, z]\, P_1'[z, v]$ and $P_2 = uw_0P_2'$, then P_1 and P_2 are edge-disjoint uv-paths in G which satisfy conditions (a) abd (b) of the lemma. This contradicts the choice of G and thus completes the proof of Lemma 3. ∎

Let V_1 be the set of vertices of degree one in a graph G. Then G is said to be *essentially 2-connected* if $G - V_1$ is 2-connected.

COROLLARY 3. *Let u and v be vertices of degree at least two in an essentially 2-connected graph G. Let P be a uv-path and put*

$$T = \{w \in N_{G-P}(\{u, v\}) \mid N(w) \subseteq V(P)\}.$$

Then G contains a cycle C such that

$$N_P(T \cup \{u, v\}) \subseteq V(C).$$

Proof. Apply Lemma 3 to the graph $G - V_1$ and put $C = P_1 \cup P_2$. ∎

Proof of Theorem 2. The proof is by contradiction. Suppose that the theorem is false. Let $G = G(a, b, k)$ be a counterexample with the minimum number of vertices and, subject to this condition, as many edges as possible. By Theorem 1, $b \geqslant 2k - 1$, and thus, since G satisfies (2),

$$a \geqslant 2k - 1. \tag{7}$$

By the minimality of $|V|$, B contains no isolated vertices. We shall show that G is essentially 2-connected. If this is not the case, then there exist a vertex v and sets of vertices M_1 and M_2 such that $V = M_1 \cup M_2$, $M_1 \cap M_2 = \{v\}$, $|(M_1 \backslash \{v\}) \cap A| \geqslant 1$, and $|(M_2 \backslash \{v\}) \cap A| \geqslant 1$. Consider the following three cases .

(i) $v \in B$. For $i = 1$ and 2, put $G_i = G(a_i, b_i, k) = G[M_i]$, $A_i =$

$A \cap V(G_i)$, and $B_i = B \cap V(G_i)$. Since $|V(G_i)| < |V|$, G_i does not satisfy the hypotheses of Theorem 2. Thus

$$b_i \geqslant \left\{ \frac{a_i}{k-1} \right\} (k-1) + 1, \tag{8}$$

and

$$\begin{aligned} b = b_1 + b_2 - 1 &\geqslant \left(\left\{ \frac{a_1}{k-1} \right\} + \left\{ \frac{a_2}{k-1} \right\} \right) (k-1) + 1 \\ &\geqslant \left\{ \frac{a}{k-1} \right\} (k-1) + 1 \end{aligned}$$

since $a = a_1 + a_2$.

(ii) $v \in A$ and $|M_1 \cap A| \equiv |M_2 \cap A| \equiv 1 \bmod (k-1)$. Choose $Q \subseteq N(v)$ such that $|Q| = k$. For $i = 1$ and 2, put $G_i = G(a_i, b_i, k) = G[M_i \cup Q]$, $A_i = A \cap V(G_i)$, and $B_i = B \cap V(G_i)$. Again, G_i does not satisfy the hypotheses of Theorem 2. Hence (8) holds, and

$$\begin{aligned} b = b_1 + b_2 - k &\geqslant \left(\left\{ \frac{a_1}{k-1} \right\} + \left\{ \frac{a_2}{k-1} \right\} \right) (k-1) + 2 - k \\ &= \left(\left\{ \frac{a_1}{k-1} \right\} + \left\{ \frac{a_2}{k-1} \right\} - 1 \right) (k-1) + 1 \\ &= \left\{ \frac{a}{k-1} \right\} (k-1) + 1 \end{aligned}$$

since $a = a_1 + a_2 - 1$ and $a_1 \equiv a_2 \equiv 1 \bmod (k-1)$.

(iii) $v \in A$ and either $|M_1 \cap A| \not\equiv 1 \bmod (k-1)$ or $|M_2 \cap A| \not\equiv 1 \bmod (k-1)$. For $i = 1$ and 2, put $G_i = G(a_i, b_i, k) = G[M_i \backslash \{v\}]$, $A_i = A \cap V(G_i)$ and $B_i = B \cap V(G_i)$. Again (8) holds, and

$$\begin{aligned} b = b_1 + b_2 &\geqslant \left(\left\{ \frac{a_1}{k-1} \right\} + \left\{ \frac{a_2}{k-1} \right\} \right) (k-1) + 2 \\ &\geqslant \left\{ \frac{a}{k-1} \right\} (k-1) + 2 \end{aligned}$$

since $a = a_1 + a_2 + 1$ and either $a_1 \not\equiv 0 \bmod (k-1)$ or $a_2 \not\equiv (k-1)$.

All cases contradict (2) and thus we may assume that G is essentially 2-connected. Let

$$P = x_1 y_1 x_2 \cdots x_{m-1} y_{m-1} x_m$$

be a path in G, chosen such that $x_i \in A$ for $1 \leqslant i \leqslant m$ and m is as large as possible. Put

$$T = N_{G-P}(\{x_1, x_m\}).$$

The proof splits into two cases which depend on the size of T.

(a) $|T| \geqslant k-1$. If $|N(T)| \leqslant k-1$ then, since $T \subseteq B$, $N(T) \subseteq A$, and $|A \backslash N(T)| \geqslant k \geqslant 2$ by (7), it can be seen that

$$H = G - (T \cup N(T))$$

satisfies the hypotheses of Theorem 2. Because $|V(H)| < |V|$, H constains a cycle of length at least $2k$, contradicting the choice of G.

Hence suppose $|N(T)| \geqslant k$. By the choice of P as a path of maximum length between vertices of A, $N(T) \subseteq V(P)$. Thus, by Corollary 3, there exists a cycle C in G such that $N(T) \subseteq V(C)$. Since $N(T) \subseteq A$, this implies that

$$|V(C)| \geqslant 2\,|N(T)| \geqslant 2k.$$

This contradicts the choice of G and completes the proof of case (a).

(b) $|T| \leqslant k-2$. For $i = 1$ and m, put $U_i = N_P(x_i)$. Clearly, we may choose vertices $x \in A$ and $y \in B$ such that x is not joined to y in G. Since G is a counterexample to Theorem 2 with as many edges as possible, x is connected to y in G by a path of length at least $2k-2$. Thus $m \geqslant k$, and no vertex of T is joined to both x_1 and x_m; otherwise, there would exist a cycle of length $2m$. Thus

$$|T| + |U_1| + |U_m| = d(x_1) + d(x_m) \geqslant 2k$$

and hence

$$|U_1| + |U_m| \geqslant 2k - |T| \geqslant k + 2. \tag{9}$$

Put

$$U_m^+ = \{y_i \in V(P) \mid y_{i-1} \in U_m\}.$$

Then $|U_m^+| = |U_m| - 1$ and, using (9),

$$|U_1| + |U_m^+| = |U_1| + |U_m| - 1 \geqslant k + 1. \tag{10}$$

Consider the following two cases which depend on the size of m.

(b_1) $m \geqslant k+1$. If there exists a vertex $y_i \in U_1 \cap U_m^+$, then

$$C = P[x_1, y_{i-1}]\; y_{i-1} x_m P[x_m, y_i]\; y_i x_1$$

is a cycle in G of length $2(m-1) \geqslant 2k$. Thus

$$U_1 \cap U_m^+ = \varnothing. \tag{11}$$

Suppose there exist vertices $y_i \in U_1$ and $y_j \in U_m$ such that $i > j$. Choosing $i-j$ as small as possible, it can be seen that the cycle

$$C' = P[x_1, y_j]\ y_j x_m P[x_m, y_i]\ y_i x_1$$

contains every vertex of $U_1 \cup (U_m^+ \backslash y_{j+1})$. Hence, by (10) and (11),

$$|B \cap V(C')| \geqslant |U_1| + |U_m^+| - 1 \geqslant k.$$

Thus $|V(C')| \geqslant 2k$, which contradicts the choice of G.

Hence we may assume that $|U_1 \cap U_m| \leqslant 1$ and thus, by (9),

$$|U_1 \cup U_m| \geqslant |U_1| + |U_m| - 1 \geqslant k+1.$$

By Corollary 3, however, G contains a cycle C'' such that $N_P(\{x_1, x_m\}) = U_1 \cup U_m \subseteq V(C'')$. Hence

$$|B \cap V(C'')| \geqslant |U_1 \cup U_m| \geqslant k+1$$

and $|V(C)| \geqslant 2(k+1)$. This contradicts the choice of G and completes the proof of subcase (b_1).

(b_2) $m = k$. Since $U_1 \cup U_m^+ \subseteq B \cap V(P)$,

$$|U_1 \cup U_m^+| \leqslant |B \cap V(P)| = k-1.$$

Thus (10) implies that $U_1 \cap U_m^+ \neq \varnothing$ and hence G contains a cycle C of length $2(k-1)$. Relabelling the vertices of P, let

$$C = x_1 y_1 x_2 y_2 \cdots x_{2k-2} y_{2k-2} x_1.$$

Suppose two vertices $x,\ x' \in A \backslash V(C)$ have a common neighbour $y \in B \backslash V(C)$. Since $G - V_1$ is 2-connected, there exist two disjoint paths from $\{x, x', y\}$ to $V(C)$. Thus there exists a path P' from $\{x, x'\}$ to $V(C)$ which is internally disjoint from $\{x, x', y\} \cup V(C)$. Clearly P' can be extended to a path containing $\{x, x'\} \cup (A \cap V(C))$. Since

$$|\{x, x'\} \cup (A \cap V(C))| = k+1,$$

this contradicts the choice of P as a path of maximum length between the vertices of A. Thus no two vertices of $A \backslash V(C)$ are joined to the same vertex of $B \backslash V(C)$.

Suppose that two vertices x_i, $x_j \in A \cap V(C)$ are joined to the same vertex $y \in B \backslash V(C)$. Then no vertex $x \in A \backslash V(C)$ can be joined to both y_i and y_j, otherwise the cycle

$$C_{2k} = y_i x_{i+1} y_{i+1} \cdots x_j y x_i y_{i-1} x_{i-1} \cdots y_j x y_i$$

would contradict the choice of G. Thus, each vertex in $A \backslash V(C)$ is joined to at most $k-2$ vertices of C and, hence, to at least two vertices of $B \backslash V(C)$. Since no two vertices of $A \backslash V(C)$ are joined to the same vertex of $B \backslash V(C)$, this implies that

$$\begin{aligned} b = |B \cap V(C)| + |B \backslash V(C)| &\geqslant k - 1 + 2\,|A \backslash V(C)| \\ &= k - 1 + 2(a - k + 1) = a + (a - k + 1) \geqslant a + k, \end{aligned}$$

by (7), contradicting (2).

Thus we may assume that no two vertices of $A \cap V(C)$ are joined to the same vertex of $B \backslash V(C)$. For $x_i \in A \cap V(C)$,

$$\varepsilon(x_i, V(C)) \leqslant k - 1,$$

and hence there exists a vertex $y(x_i) \in N_{G-C}(x_i)$. Put

$$W = \{\, y(x_i) \mid x_i \in A \cap V(C) \}.$$

Then

$$|W| = |A \cap V(C)| = k - 1.$$

For $x \in A \backslash V(C)$, put

$$Z(x) = \{x_i \in A \cap V(C) \mid y(x_i) \in N(x)\}$$

and

$$Z(x)^+ = \{\, y_i \mid x_i \in Z(x) \}.$$

Then x is not joined to any vertex of $Z(x)^+$, otherwise we again obtain a cycle of length $2k$. Thus

$$\varepsilon(x, V(C)) \leqslant k - 1 - |Z(x)^+| = k - 1 - |Z(x)|,$$

and, since $\varepsilon(x, W) = |Z(x)|$,

$$\varepsilon(x, V(C) \cup W) \leqslant k - 1.$$

Hence, for each $x \in A \backslash V(C)$, there exists a vertex $y(x) \in N(x) \backslash (V(C) \cup W)$.

Moreover, $y(x) \neq y(x')$ if $x \neq x'$ since no two vertices of $A \backslash V(C)$ are joined to the same vertex of $B \backslash V(C)$. Thus

$$\begin{aligned} b &\geqslant |B \cap V(C)| + |W| + |\{y(x) \mid x \in A \backslash V(C)\}| \\ &= (k-1) + (k-1) + (a-k+1) = a+k-1. \end{aligned}$$

Again this contradicts (2) and completes the proof of Theorem 2. ▮

It seems likely that the bounds on b in Theorems 1 and 2 can be increased under the added condition that $G(a, b, k)$ be 2-connected.

Conjecture. Suppose that a graph $G(a, b, k)$ is 2-connected and satisfies

$$\begin{aligned} b &\leqslant 3(k-2)+1 & &\text{if} \quad a \leqslant k \\ &\leqslant \left\{\frac{2(a-\varepsilon_k)}{k-1-\varepsilon_k}\right\}(k-2)+1 & &\text{if} \quad a \geqslant k, \end{aligned}$$

where ε_k is equal to one if k is even, and zero if k is odd. Then $G(a, b, k)$ contains a cycle of length at least $2\min(a, k)$.

If true, the conjecture would be best possible, as can be seen from the following example. Let a and k be integers such that $a \geqslant k \geqslant 2$. Put

$$2(a-\varepsilon_k) = q(k-1-\varepsilon_k) + r$$

for integers q and r, $0 < r \leqslant k-1-\varepsilon_k$. Let $G(a,b,k)$ be the 2-connected graph formed from $q+1$ disjoint complete bipartite graphs $K_{a_i,k-2}$, for $i \in \{1, 2, \ldots, q+1\}$, by joining two extra vertices b_1 and b_2 to every vertex in the a_i-set of each $K_{a_i,k-2}$. Putting $a_1 = \frac{1}{2}(k-1+\varepsilon_k)$, $a_i = \frac{1}{2}(k-1-\varepsilon_k)$ for $i \in \{2, 3, \ldots, q\}$, and $a_{q+1} = \frac{1}{2}r$, it follows that $G(a, b, k$ satisfies

$$b = \left\{\frac{2(a-\varepsilon_k)}{k-1-\varepsilon_k}\right\}(k-2)+2$$

and contains no cycle of length greater than or equal to $2k$.

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